Kuwait University	Math 101	Date:	June 4, 2007
Dept. of Math. & Comp. Sci.	Final Exam	Duration:	Two Hours

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions. Each question weighs 4 points.

1. Evaluate $\lim_{x\to 0} (\csc x - \cot x)$, if it exists.

2. Find the $x-$coordinates of the points at which the function f is discontinuous, where

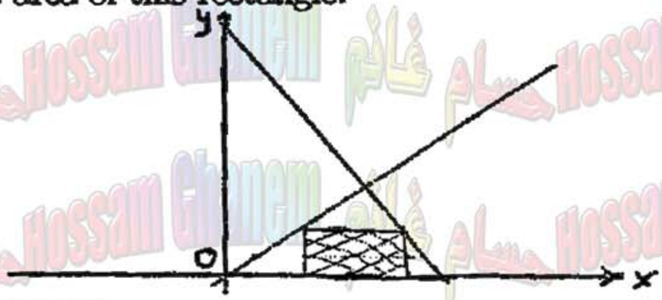
$$f(x) = \frac{x^2 - x}{|x|(x^2-1)}.$$

Classify the types of discontinuity of f as removable, jump, or infinite.

3. Let $f(x) = x^{\frac{1}{3}}(x^2-3)^{\frac{1}{3}}$. Find the x-coordinate of the points at which the tangent line to the graph of f is horizontal and the x-coordinate of the points at which the tangent line to the graph of f is vertical.

4. Find $\frac{dy}{dx}$, where $y = x\left[\sin\left(\frac{x^2+1}{x+1}\right)\right]^3$.

5. Let T be the triangle bounded by the lines: $y=0$, $y=3x$ and $y=30-2x$. A rectangle with one side lying on the $x-$axis is inscribed inside T, as shown in the figure. What is the largest possible area of this rectangle.



6. Evaluate: $\int \frac{\sqrt{u^6+5u^4}}{u}\,du.$

7. Evaluate: $\int_0^2 x(x^2-2)^5\,dx.$

8. Let $f(x) = \int_1^{x^3} \cos^3 u\,du + \int_1^{x^3+x} \sqrt{1+s^4}\,ds + \int_{x^3}^4 \cos^3 u\,du$. Show that f is an increasing function.

9. Find the area of the region bounded by $y=\sqrt{x}-2$ and the $x-$axis, from $x=0$ to $x=9$

10. The region bounded by the curves $x=y^2$ and $x=y^3$ is revolved about:

    (a) the line $x=5$,

    (b) the line $y=-3$.

    Set up an integral that can be used to find the volume of the resulting solid in each case.

1. $\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\frac{1-\cos x}{x}}{\frac{\sin x}{x}}\right) = \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{0}{1} = \boxed{0}$.

2. f is not continuous at $x = 0, x = 1$ and $x = -1$ (f is undefined).

$\lim_{x\to -1^\pm} f(x) = \lim_{x\to -1^\pm}\frac{x^2 - x}{|x|(x-1)(x+1)} = \boxed{\mp\infty} \implies f$ has a *infinite* discontinuity at $x = -1$.

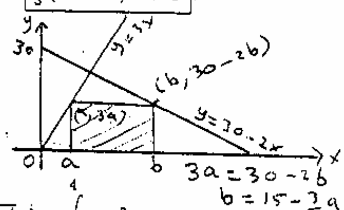
$\lim_{x\to 0^\pm} f(x) = \lim_{x\to 0^\pm}\frac{x(x-1)}{|x|(x-1)(x+1)} = \lim_{x\to 0^\pm}\frac{x(x-1)}{\pm x(x-1)(x+1)} = \boxed{\pm 1} \implies f$ has a *jump* discontinuity at $x = 0$.

$x = 0. \lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x(x-1)}{|x|(x-1)(x+1)} = \boxed{\frac{1}{2}} \implies f$ has a *removable* discontinuity at $x = 1$.

3. $f(x) = x^{\frac{1}{3}}(x^2-3)^{\frac{1}{3}} = (x^3-3x)^{\frac{1}{3}} \implies \boxed{f'(x) = \frac{3x^2-3}{3(x^3-3x)^{\frac{2}{3}}}}$ f has horizontal tangent when $f'(x) = 0$, i.e., when $3x^2 - 3 = 0 \implies f$ has horizontal tangent at $\boxed{x = -1 \text{ and } x = 1.}$. Since f is continuous, then f has vertical tangent at $f'(x)$ is undefined, i.e., when $3(x^3-3x)^{\frac{2}{3}} = 0 \implies f$ has vertical tangent at $\boxed{x = -\sqrt{3}, x = 0, x = \sqrt{3}}$.

4. $$\frac{dy}{dx} = \sin^3\left(\frac{x^2+1}{x+1}\right) + 3\,x\,\sin^2\left(\frac{x^2+1}{x+1}\right)\cos\left(\frac{x^2+1}{x+1}\right)\left(\frac{x^2+2x-1}{(x+1)^2}\right).$$

5. $A = 3a(b-a) = 3a\frac{30-3a-2a}{2} = \frac{15}{2}(6a - a^2) \implies \frac{dA}{da} = 15(3-a)$ & $\frac{dA}{da} = 0 \implies \boxed{a = 3}$. $\left.\frac{d^2A}{da^2}\right|_{a=3} < 0 \implies A$ is maximum at $a = 3$. $\implies A_{\max} = A(3) = \boxed{\frac{135}{2} = 67\frac{1}{2}}$.



6. $\frac{\sqrt{u^6+5u^4}}{u} = \frac{\sqrt{u^4(u^2+5)}}{u} = \frac{u^2\sqrt{u^2+5}}{u}$. Put $x = u^2 + 5 \implies dx = 2u\,du$

$$\int\frac{\sqrt{u^6+5u^4}}{u}\,du = \int\frac{u^2\sqrt{u^2+5}}{u}\,du = \frac{1}{2}\int\sqrt{x}\,dx = \frac{1}{3}x^{\frac{3}{2}} + C = \boxed{\frac{1}{3}(u^2+5)^{\frac{3}{2}} + C}.$$

7. Put $u = x^2 - 2 \implies du = 2x\,dx$ & $u(0) = -2$, $u(2) = 2$.

$$\int_0^2 x(x^2-2)^5\,dx = \frac{1}{2}\int_{-2}^{2} u^5\,du = 0 \text{ (u^5 is an odd function).}$$

8. $$f(x) = \int_1^{x^3}\cos^3 u\,du + \int_1^{x^3+x}\sqrt{1+s^4}\,ds + \int_{x^3}^{4}\cos^3 u\,du = \int_1^{x^3+x}\sqrt{1+s^4}\,ds + \int_1^{4}\cos^3 u\,du \implies$$

$\boxed{f'(x) = (3x^2+1)\sqrt{1+(x^3+x)^4} > 0} \implies f$ is an increasing function.

9. $$Area = A_1 + A_2 = \int_0^4 -(\sqrt{x}-2)\,dx + \int_4^9(\sqrt{x}-2)\,dx = \left.\tfrac{2}{3}x^{\frac{3}{2}} - 2x\right]_4^0 + \left.\tfrac{2}{3}x^{\frac{3}{2}} - 2x\right]_4^9 = \boxed{\frac{16}{3}}.$$

10. (a) REVOLUTION ABOUT THE LINE $x = 5$:

$$Volume = \pi\int_0^1\left[(5-y^3)^2 - (5-y^2)^2\right]dy \text{ OR } Volume = 2\pi\int_0^1(5-x)(\sqrt[3]{x} - \sqrt{x})\,dx.$$

(b) REVOLUTION ABOUT THE LINE $y = -3$:

$$Volume = 2\pi\int_0^1(y^2-y^3)\,[y-(-3)]\,dy \text{ OR } Volume = \pi\int_0^1\left[(\sqrt[3]{x}+3)^2 - (\sqrt{x}+3)^2\right]dx.$$